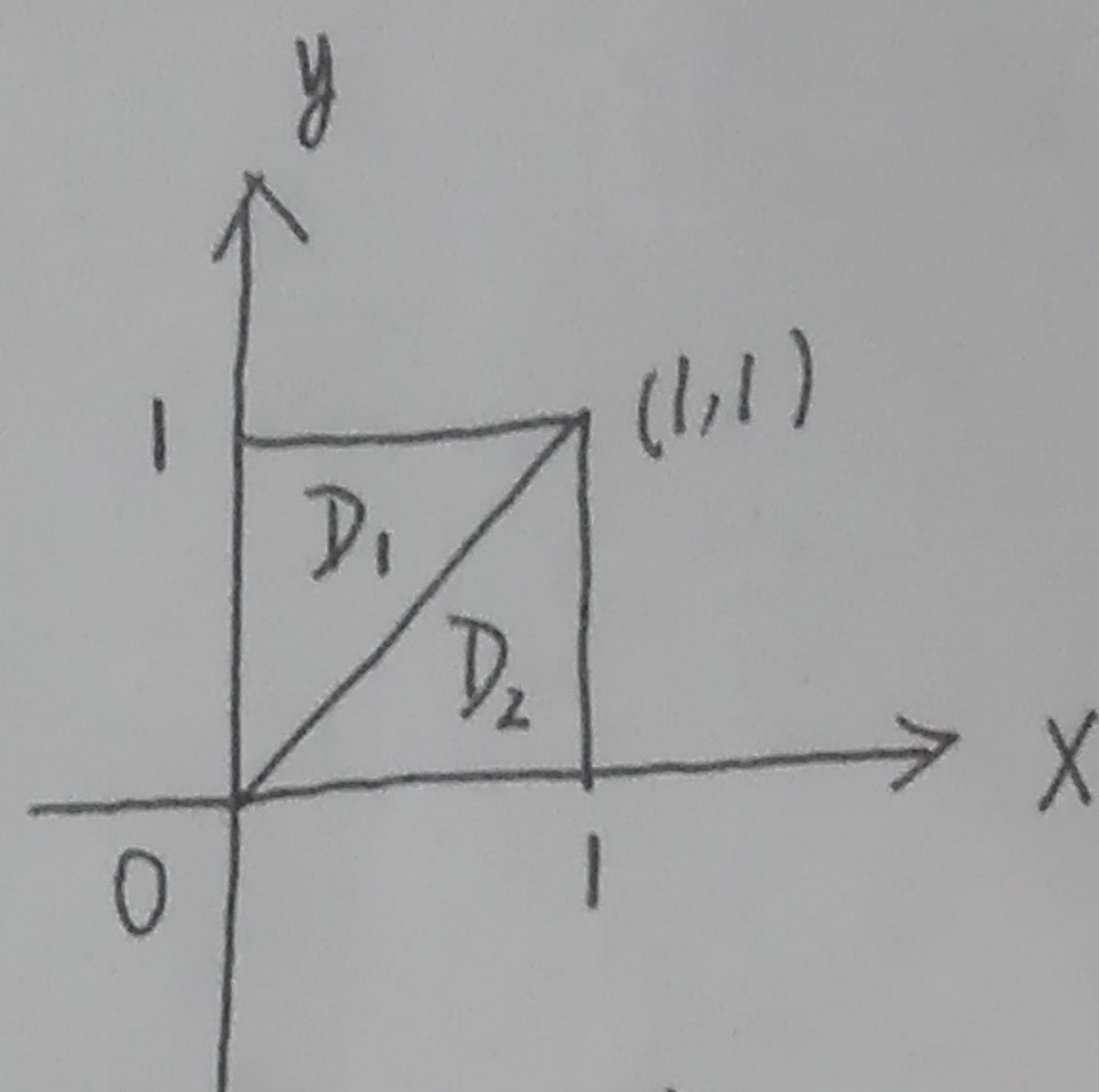


9. 设函数 $f(x)$ 在 $[0, 1]$ 上连续, 且 $\int_0^1 f(x) dx = A$, 求 $\int_0^1 dx \int_x^1 f(x)f(y) dy$

解: 如右图 $D: 0 \leq x \leq 1 \quad 0 \leq y \leq 1$

$$D_1: 0 \leq x \leq 1 \quad x \leq y \leq 1$$

$$D_2: 0 \leq x \leq 1 \quad 0 \leq y \leq x$$



显然 $D = D_1 + D_2$ 且 D_1 与 D_2 关于

直线 $y=x$ 对称。点 (x, y) 关于 $y=x$ 的对称点为 (y, x)

$$\text{设 } f(x, y) = f(x) \cdot f(y) = f(y) \cdot f(x) = f(y, x)$$

即函数 $f(x) \cdot f(y)$ 关于 $y=x$ 的对称区域为偶函数,

$$\text{于是 } \int_0^1 dx \int_x^1 f(x)f(y) dy = \iint_{D_1} f(x)f(y) d\sigma = \iint_{D_2} f(x)f(y) d\sigma$$

$$= \frac{1}{2} \iint_D f(x)f(y) d\sigma$$

$$= \frac{1}{2} \int_0^1 dx \int_0^1 f(x)f(y) dy$$

$$= \frac{1}{2} \int_0^1 f(x) dx \int_0^1 f(y) dy = \frac{1}{2} A \cdot A = \frac{A^2}{2}$$

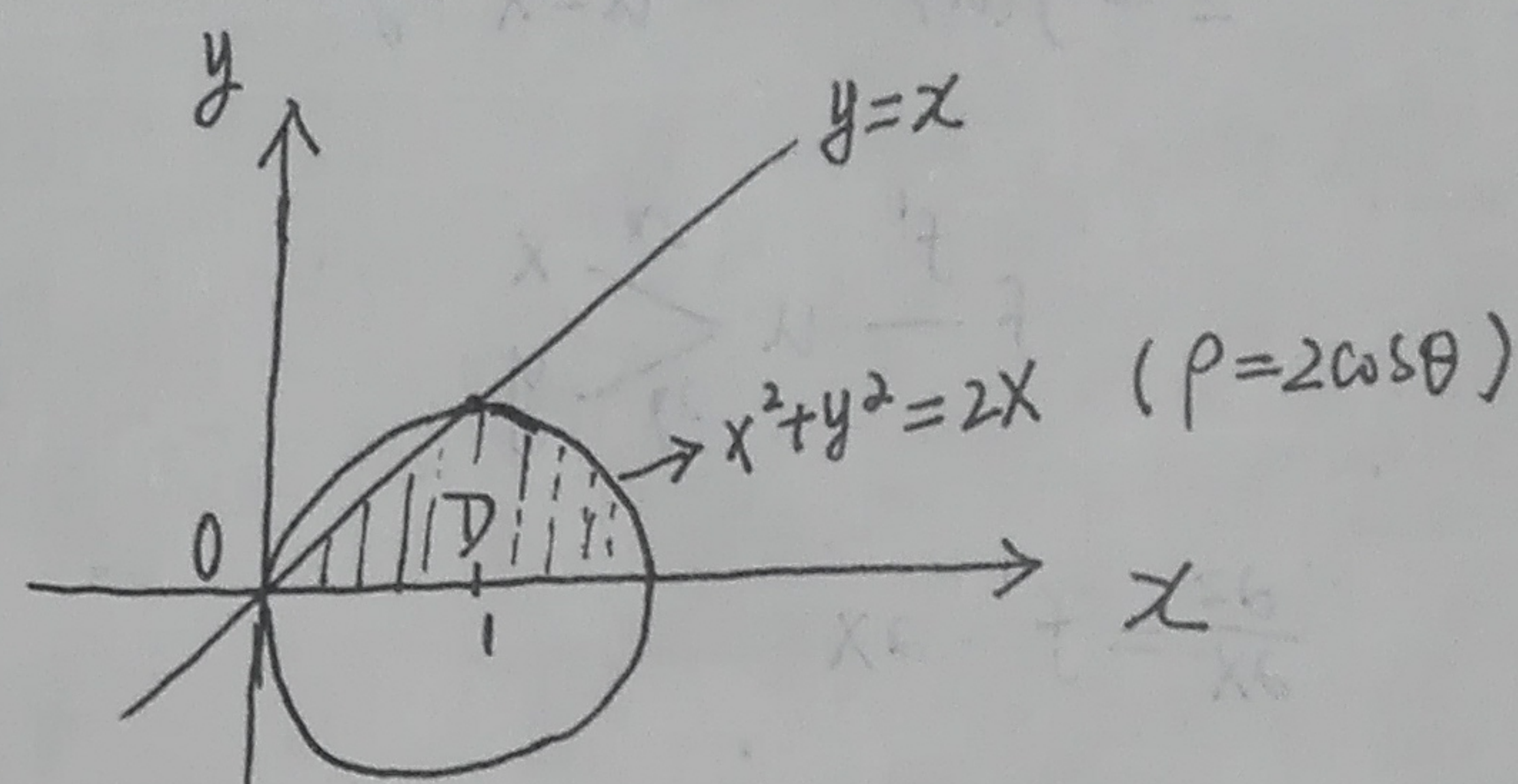
P195 11 求二重积分 $\iint_D \sqrt{x^2+y^2} dx dy$, 其中 $D = \{(x,y) \mid 0 \leq y \leq x, x^2+y^2 \leq 2x\}$

解: 用极坐标计算

$x^2+y^2=2x$ 化为极坐标系方程

$$(\rho \cos \theta)^2 + (\rho \sin \theta)^2 = 2 \cdot \rho \cos \theta \quad \text{即}$$

$$\rho = 2 \cos \theta$$



$$D: \quad 0 \leq \theta \leq \frac{\pi}{4} \quad 0 \leq \rho \leq 2 \cos \theta$$

$$\iint_D \sqrt{x^2+y^2} dx dy = \int_0^{\frac{\pi}{4}} d\theta \int_0^{2 \cos \theta} \rho \cdot \rho d\rho = \int_0^{\frac{\pi}{4}} \left(\frac{1}{3} \rho^3 \Big|_{\rho=0}^{2 \cos \theta} \right) d\theta$$

$$= \frac{8}{3} \int_0^{\frac{\pi}{4}} \cos^3 \theta d\theta = \frac{8}{3} \left(\sin \theta - \frac{\sin^3 \theta}{3} \right) \Big|_0^{\frac{\pi}{4}}$$

~~$$= \frac{8}{3} \left(\frac{\sqrt{2}}{2} - \frac{(\frac{\sqrt{2}}{2})^3}{3} \right) = \frac{8}{3} \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} \right) = \frac{10}{9} \sqrt{2}$$~~

$$= \frac{8}{3} \left(\frac{\sqrt{2}}{2} - \frac{(\frac{\sqrt{2}}{2})^3}{3} \right) = \frac{8}{3} \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} \right) = \frac{10}{9} \sqrt{2}$$

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求二重积分 $\iint_D \frac{x^2}{x^2+y^2} d\sigma$, 其中 $D = \{(x,y) \mid x^2+y^2 \leq 1\}$

方法一 用极坐标计算

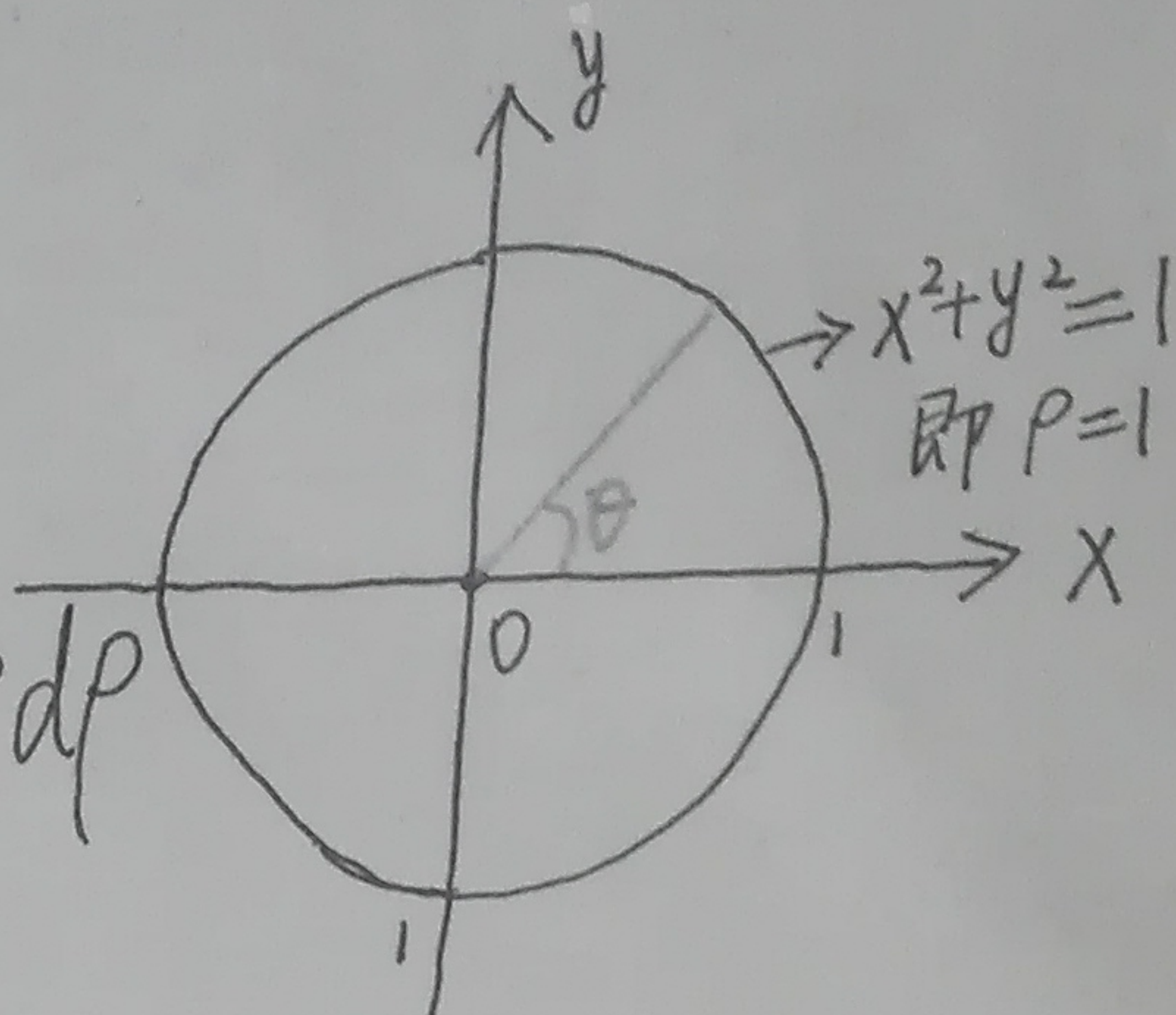
$$D: 0 \leq \theta \leq 2\pi \quad 0 \leq \rho \leq 1$$

$$\text{原式} = \int_0^{2\pi} d\theta \int_0^1 \frac{(\rho \cos \theta)^2}{(\rho \cos \theta)^2 + (\rho \sin \theta)^2} \cdot \rho d\rho$$

$$= \int_0^{2\pi} d\theta \int_0^1 \cos^2 \theta \rho d\rho$$

$$= \int_0^{2\pi} \cos^2 \theta \cdot \frac{1}{2} d\theta = \frac{1}{2} \int_0^{2\pi} \cos^2 \theta d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \frac{1}{2} d\theta = \frac{\pi}{2}$$



方法二. 用性质

$$\iint_D \frac{x^2}{x^2+y^2} d\sigma = \iint_D \frac{y^2}{y^2+x^2} d\sigma$$

$$= \frac{1}{2} \iint_D \left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} \right) d\sigma$$

$$= \frac{1}{2} \iint_D d\sigma$$

$$= \frac{1}{2} S_D = \frac{\pi}{2}$$